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Some Conics with Names

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SOME CONICS WITH NAMES

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Scattered through the field of elementary mathematics there are a number of conics which have received names. It is the purpose of this paper to group them together for reference. In the list that follows, the most common method of defining the curve is given and the given references are not necessarily the earliest reference. Articles have been selected as far as possible because they give a good discussion of the curve.

It seems wise to omit the circles in this list because of their number. These conics have been grouped around certain concepts and properties.

"Normals"

(1) The Hyperbola of Apollonius

In the plane of a fixed conic with center C, take a fixed point O. Draw a perpendicular OP from O to a variable diameter CP, of the conic. Produce this perpendicular to meet the conjugate diameter of CP in a point D. The locus of D is a rectangular hyperbola passing through D and C. Its asymptotes are parallel to the axes of the given conic. This hyperbola intersects the given conic in the feet of the four normals which can be drawn from O to the conic.

Apollonius in his Conics, Lib. V., props. 58-63 uses this hyperbola for drawing the normals from a point to a given conic. See Taylor, Ancient and Modern Geometry of Conics, p. 265.

(2) The Parabola of Chasles

Chasles, in his Sectiones Coniques, p. 145, replaces the hyperbola of Apollonius by a parabola for drawing the normals from a given point to a conic. Consider an ellipse E with center O, foci F and F′, and axes along the coördinate axes Ox and Oy. Let P be any point in the plane of the ellipse. Draw a variable secant through P cutting E in M and N. From R the pole of MN drop a perpendicular to MN. The envelope of these perpendiculars is Chasles' parabola. Since there are eighteen notable lines tangent to this conic, it is sometimes called "The Parabola of Eighteen Lines."

It is not difficult to pick out some of these lines. They are the four tangents at the feet of the four normals from P to the ellipse; the axes Ox and Oy; the normals to E at A and B, the points of contact of the tangents from P to E; the line AB and its perpendicular bisector; the perpendiculars erected at F and F′ to the lines PF and PF′; and the bisectors of the angles formed by the lines PA and PB.

The remaining four tangents of note and a large number of beautiful theorems concerning this locus are set forth in an article by M. F. Balitrand, Nouvelles Annales de Matnematique (N. A. M.) 1913, p. 198.

(3) The Conics of Thomson and Darboux

The conics inscribed in a triangle such that the normals at the points of contact with the sides pass through a point Q are known as the conics of Thomson.

The conics circumscribed to a triangle such that the normals at the vertices pass through a point P are known as the conics of Darboux.

If the same triangle is used the points P and Q run over the same locus, a cubic known under two names, the cubic of Darboux and the seventeen-point cubic.

The triangles formed by the points of contact of the conics of Thomson with the sides of the triangle and the triangles formed by the tangents to the conics of Darboux at the vertices of the triangle are homologous to the fixed triangle. The locus of the center of homology is the cubic of Lucas.

The locus of the centers of the conics of Thomson is Thomson's cubic. See N.A. M., 1866, p. 95, 420; 1876, p. 240, 550; 1880, p. 144; 1881, p. 520.

(4) Purser's Parabola

Consider a normal to an ellipse, say at the fixed point P′. From every point on this normal, three additional normals can be drawn to the ellipse. The sides of the triangle formed by their three feet touch a parabola, called Purser's parabola. See Quarterly Journal VII. 1867, p. 66.

"The Kiepert Configuration"

On the sides of a given triangle, isosceles triangles of variable base angle Θ are constructed with their vertices all drawn inwardly or outwardly. The vertices of these triangles form a new triangle A′B′C′ called Kiepert's triangle. The triangles ABC *and A′B′C′* are in perspective. Let the center of perspectivity be P and let the axis of homology be p. The following loci are associated with this configuration.

(a) The locus of P is Kiepert's hyperbola. It is an equilateral hyperbola and passes through the vertices A,B,C and through the center of gravity G, and through the orthocenter H and through Tarry's point. See N. A. M. 1869, p. 41.

(b) The parabola of Kiepert is the envelope of p. It is inscribed in the triangle ABC. It touches the line of Lemoine. It was studied by Artzt who also obtained the following group of parabolas.

(c) The envelopes of the sides of the triangle of Kiepert are three parabolas known as the parabolas of Artzt (second group). Their foci are the vertices of the second triangle of Brocard and their directrices are the medians of the triangle ABC. See Programme du Gymnase de Rechlinghausen, 1884-1886. This reference is not available to the author. See also Mathesis, 1895, p. 10.

"ISOGONAL TRANSFORMATION"

If x,y,z are the actual perpendicular distances from the sides of the reference triangle ABC to the point P, they are called the normal coödinates of the point P. If their three ratios x : y : z are given the actual coördinates can be found from the relation $ax + by + cz = 2\Delta$ where Δ is the area of the triangle ABC and a, b, c are the lengths of the sides. The point whose coördinates are given by the three ratios $1/x : 1/y : 1/z$ is called the isogonal conjugate of P.

If we replace x, y, z by $1/x$, $1/y$ and $1/z$ respectively in the equation of any curve, the curve is transformed point by point into another curve whose order is in general double the order of the original. For example all the lines in the plane are transformed into conics circumscribing the triangle ABC. From this theory it is well known that all the diameters of the circumcircle are transformed into rectangular hyperbolas which pass through the orthocenter H. The following lines are transformed into conics of note.

(a) The diameter OK, K being the symmedian point of the triangle, is transformed into Kiepert's hyperbola. See N. A. M., 1869, p. 41.

(b) The diameter OH is transformed into the hyperbola of Jerabek. The line OH is known as Euler's line. This curve is also known as the "Hyperbola Γ." See Mathesis, 1888, p. 81.

(c) The diameter OI, I being the incenter, is transformed into the hyperbola of Feuerbach. Its center is at the point of contact of the incircle and the nine-point circle. See Mathesis, 1893, p. 81; 1903, p. 265; 1905, p. 177.

(d) The directrices of Brocard's ellipse become hyperbolas known as Neuberg's hyperbolas and also by the name of Simson's conics. They circumscribe the triangle of reference and pass through Tarry's point. See Mathesis, VI, p. 5-7. See also Vigarie, Bibliography of the Triangle, 1889.

"Inscribed Conics"

If a conic is inscribed in the triangle of reference ABC, its foci are isogonal conjugates of one another. Some of the following conics have their foci at a pair of remarkable points of the triangle ABC.

(a) The Brocard ellipse has its foci at the Brocard points of the reference triangle and touches its sides at the feet of the symmedians. See The Companion to the Weekly Problem Papers, by J. J. Milne, 1888, p. 104, 111.

(b) The ellipse of Lemoine has its foci at the center of gravity G and at the Lemoine point K. It touches the sides of the reference triangle at the feet of the symmedians of the triangles BGC, CGA and AGB.

(c) The ellipse of Simmon has its foci at the first isodynamic point and the first isogonal center of the triangle ABC. It touches the sides of the triangle at the feet of the lines joining the vertices to the first isogonal center. See Proceedings of the London Mathematical Society, XVIII, p. 418.

(d) The ellipse of Nagel touches the sides of the triangle ABC in the points D_a, D_b, D_c, where D_a is the point of contact of the excircle with the side a of the triangle, etc. The circle through these three points is Nagel's circle. The center of this ellipse is on Feuerbach's hyperbola and it passes through Feuerbach's point. The foci are not significant. See Mandart, Mathesis, 1922, p. 125.

(e) The ellipse of Euler has its foci at the circumcenter and the orthocenter. The focal axis is of length R, where R is the radius of the circumcircle. See Mathesis, 1915, p. 20; 1922, p. 183.

(f) In addition to the ellipse of Euler there are three hyperbolas H_a, H_b, and H_c known as the conics of Euler. The hyperbola H_a is tangent to the lines BC, CH, and BH, with foci at H and O_a, where O_a is a point symmetric to O with respect to the side a of the triangle, etc. Since they are inscribed in special triangles, their foci are on little interest.

"The Anticomplementary Transformation"

If the point x, y, z is transformed into the point $-x+y-+z$, x − y + z, x + y − z, its transform is called the point anticomplementary of the first. The following conics arise from this transformation.

(a) The Brocard parabolas are obtained from Artzt's parabolas (both groups) by this transformation. See Memoirs of the Academy of Montpellier, 1886. This reference is not accessible to the author.

(b) If this type of transformation is applied to the circumcircle, it is transformed into an ellipse with center I (incenter) and passing through the feet of the internal bisectors of the angles of the triangle. This ellipse is known as the ellipse of De Longchamps. See Association Francaise pour L'Avancement des Sciences, 1886, p. 69.

(c) If M be any point in the plane and if AM cut BC in the point A′, etc., then there exists a conic with center M passing through the points A′, B′, C′. This is a generalization of the above and the conic is said to be centrally associated with M. It is usually written Conic (M). Hence references to conics such as "Conic (I)," "Conic (J)," and "Conic (K) are now evident.

"Harmonic System of Conics"

Consider a complete quadrilateral. Let w, w′ be the two points on any side such that each with the three points on that side form an equianharmonic system of points. Let i, i′ be the double points of the involution determined upon any diagonal by the two opposite vertices and by the intersections of the other two diagonals. The four pairs of points w, w′ and the three pairs i, i′ lie on a conic, known as the *fourteen-point conic*. See Messenger of Mathematics, 3. 1886, p. 13, 68.

The locus of a point such that the four tangents drawn from it to two conics form a harmonic pencil is known as *Salmon's Conic* or *Conic F*. Since two conics intersect in four points, it is not difficult to show that Salmon's conic is the fourteen-point conic of the four points of intersection. See Quarterly Journal of Mathematics, 7. 1866, p. 20.

If three conics are so related that each is the polar reciprocal of any other with respect to the third, they are said to form a harmonic system. In this case each conic is the Salmon conic of the other two. The Brocard ellipse and the Jerabek hyperbola and their Salmon conic form a harmonic system.

"Miscellaneous Conics"

(a) The centers of all the conics on four points lie on a conic known as the "Eleven-point Conic." The eleven points are the midpoints of the six sides of the complete quadrangle determined by the four points, their three diagonal points, and the two points of contact of the two parabolas in the pencil with the line at infinity. If the two base conics of the pencil are rectangular hyperbolas, the center locus is a circle known as the "Nine-points Circle." It passes through the circular points on the line at infinity. See Taylor, Ancient and Modern Geometry of Conics, p. 284.

(b) The "*Conic GK*" is the locus of centers of all the conics which pass through the quadrangle A B C K. This conic cuts the circle of Euler in the center of the hyperbola of Jerabek. See Mathesis, 1905, p. 148, 170.

(c) The *ellipse of Steiner* is an ellipse that circumscribes the triangle of reference in such a way that the tangents at the vertices are parallel to the opposite side. The center of the ellipse is the centroid of the triangle. The triangle is a triangle of maximum area inscribed in the ellipse. This configuration has a large number of properties listed in Mathesis, 1895, p. 42, 81; 1897, p. 88. See also Steiner's Gesammelte Werke, t. II. p. 691.

(d) Let A′ B′ C′ be points on the side of a the triangle ABC such that the following ratios exist: AC′/AB = BA′/BC = CB′/CA. The triangles ABC and A′B′C′ are homologous and have the same Brocard angle. Their center of perspective generates Brocard's circle. The envelopes of their sides are parabolas known as the first group of Artzt. See reference cited above.

(e) If a curve C rolls on a second curve C_0 without slipping, every point associated with C derives a curve. If the common point of the two curves C and C_0 is taken as the pole and the common tangent as the polar axis, then it is possible to determine the coördinates of any point M in the plane of C in terms of the various radii of the two given curves and the angle Θ that is made by joining M to the pole with the initial line. The locus of poles of any straight line associated with the curve C is a conic known as the *conic of Rivals*. See Mathesis, 1931, p. 364.

(f) The envelope of the polar lines of the points of the line GK with respect to the triangle ABC is a parabola. This *parabola is Neuberg's*. It is inscribed in the triangle and has the pole of the line OK with regard to the triangle as its focus. Its directrix is the line HG. See Mathesis, 1895, p. 6.

(g) The *ellipse of Fagnano* is characterized by the fact that its eccentricity is one-half the square root of two. The circle over the minor axis as diameter passes through the foci. See Sommerville, Analytical Conics, p. 128.

I have no information concerning the *conics of Apollonius* or the *parabola of Apollonius.*

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